Applications of commuting difference operators to orthogonal polynomials in several variables*

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Elementary properties of the Koornwinder-Macdonald multivariable Askey-Wilson polynomials are discussed. Studied are the orthogonality, the difference equations, the recurrence relations, and the orthonormalization constants for these polynomials. Essential in our approach are certain commuting difference operators simultaneously diagonalized by the polynomials.

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I. INTRODUCTION

It is well-known that large part of the classical theory of hypergeometric orthogonal polynomials is intimately connected with the representation theory of (simple, rank one) Lie groups and that, similarly, *basic* hypergeometric orthogonal polynomials are connected with (the representation theory of) quantum groups [17]. Moreover, it turns out that these relations with representation theory provide a fruitful framework for generalizing such polynomials to several variables [18].

Around ten years ago, Askey and Wilson introduced a now famous family of basic hypergeometric orthogonal polynomials in one variable, which contains many of the (basic) hypergeometric orthogonal polynomials studied in the literature as special (limiting) cases [1,9]. Recently, a multivariable generalization of the Askey-Wilson polynomials was found, first for special parameters by Macdonald [11] and then by Koornwinder [10] for general parameters. It turns out that from the viewpoint of representation theory (and for special values of the parameters) the Koornwinder-Macdonald multivariable Askey-Wilson polynomials correspond to zonal spherical functions on quantum symmetric spaces of classical type [15]. Furthermore, the polynomials may be interpreted physically as the eigenfunctions for a Ruijsenaars type difference version of the Calogero-Sutherland n-particle model related to the root system BC_n [5,16].

Here we will study some elementary properties of these multivariable Askey-Wilson polynomials, with the aim of generalizing part of the well-known results by Askey and Wilson to the case of several variables. Attention will be mainly focussed on describing the difference equations, the recurrence relations, and the orthonormalization constants for the polynomials. An essential role in our discussion is played by a family of previously introduced commuting difference operators that are simultaneously diagonalized by the polynomials. Most results are stated without complete proof. A more detailed treatment including proofs can be found in [6].

II. MULTIVARIABLE ASKEY-WILSON POLYNOMIALS

A natural basis for the algebra of permutation invariant and even trigonometric polynomials is given by the monomial symmetric functions

$$m_\lambda(x) \;=\; \sum_{\lambda' \in W\lambda} e^{\alpha \sum_{j=1}^n \lambda'_j x_j}, \qquad \lambda \in \Lambda = \{ \lambda \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \}, \tag{2.1}$$

where the summation is over the orbit of λ under the action of the (Weyl) group $W(\cong S_n \ltimes (\mathbb{Z}_2)^n)$ generated by permutations and sign flips of the vector components $\lambda_1, \ldots, \lambda_n$. One can (partially) order the basis of monomial symmetric functions by defining for all $\mu, \lambda \in \Lambda$

$$\mu \leq \lambda \qquad \text{iff} \qquad \sum_{1\leq j\leq m} \mu_j \leq \sum_{1\leq j\leq m} \lambda_j \quad \text{for} \quad m = 1, \ldots, n \tag{2.2}$$

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(and $\mu<\lambda$ iff $\mu\leq\lambda$ and $\mu\neq\lambda$).

The Koornwinder-Macdonald multivariable Askey-Wilson polynomials $p_\lambda(x)$ (with $\lambda\in\Lambda$) are now defined as the (unique) trigonometric polynomials satisfying

i. $p_\lambda(x)=m_\lambda(x)+\sum_{\mu\in\Lambda,\mu<\lambda} c_{\lambda,\mu}\, m_\mu(x), \qquad c_{\lambda,\mu}\in\mathbb{C};$

ii. $\langle p_\lambda, m_\mu\rangle_\Delta=0 \quad \text{if} \quad \mu<\lambda.$

Here $\langle\cdot,\cdot\rangle_\Delta$ denotes the L^2 inner product

$$\langle m_\lambda, m_\mu\rangle_\Delta=\left(\frac{\alpha}{2\pi}\right)^n\int_{-\pi/\alpha}^{\pi/\alpha}\!\!\cdots\int_{-\pi/\alpha}^{\pi/\alpha} m_\lambda(ix)\,\overline{m_\mu(ix)}\,\Delta(ix)\,dx_1\cdots dx_n \tag{2.3}$$

with weight function

$$\Delta(x)\;=\;\Delta^+(x)\Delta^+(-x), \tag{2.4}$$

where

$$\Delta^+(x)=\prod_{1\leq j<k\leq n} d_v^+(x_j+x_k)\,d_v^+(x_j-x_k)\prod_{1\leq j\leq n} d_w^+(x_j), \tag{2.5}$$

$$d_v^+(z)=q^{-gz/2}\frac{(q^z;q)_\infty}{(q^gq^z;q)_\infty}, \qquad\qquad q=e^{-\alpha},$$

$$d_w^+(z)=q^{-(g_0+\cdots+g_3)z/2}\frac{(q^z,\,-q^z,\,q^{1/2}q^z,\,-q^{1/2}q^z;\,q)_\infty}{(q^{g_0}q^z,\,-q^{g_1}q^z,\,q^{(g_2+1/2)}q^z,\,-q^{(g_3+1/2)}q^z;\,q)_\infty}.$$

(As usual $(a;q)_\infty\equiv\prod_{l=0}^\infty(1-aq^l)$ and $(a_1,\ldots,a_k;q)_\infty\equiv(a_1;q)_\infty\cdots(a_k;q)_\infty$.) To ensure the convergence of the infinite products in Δ (2.4) it will be assumed that α is positive (so $0<q<1$); in addition, we will also assume that $g,g_r\geq 0$, $r=0,1,2,3$.

III. DIFFERENCE EQUATIONS

In [4] (see also [6, Section 7.1]) it was shown that the polynomials $p_\lambda(x)$ satisfy a system of difference equations having the structure of eigenvalue equations

$$(D_r\,p_\lambda)(x)=E_r(\rho+\lambda)\,p_\lambda(x) \tag{3.1}$$

for n independent commuting difference operators $D_1,\ldots,D_n$ of the form

$$D_r=\sum_{\substack{J\subset\{1,\ldots,n\},\,0\leq|J|\leq r\\ \varepsilon_j=\pm1,\ j\in J}} U_{J^c,\,r-|J|}(x)\,V_{\varepsilon J,\,J^c}(x)\,T_{\varepsilon J}, \qquad r=1,\ldots,n, \tag{3.2}$$

with $(T_{\varepsilon J}f)(x)=f(x+e_{\varepsilon J})$ where $e_{\varepsilon J}=\sum_{j\in J}\varepsilon_je_j$. (Here e_j denotes the jth unit vector in the standard basis of $\mathbb{R}^n$.) The coefficients of the difference operators read explicitly

$$\begin{aligned}V_{\varepsilon J,\,K}(x)=&\prod_{j\in J}w(\varepsilon_jx_j)\prod_{\substack{j,j'\in J\\ j<j'}}v(\varepsilon_jx_j+\varepsilon_{j'}x_{j'})v(\varepsilon_jx_j+\varepsilon_{j'}x_{j'}+1)\\ &\times\prod_{\substack{j\in J\\ k\in K}}v(\varepsilon_jx_j+x_k)v(\varepsilon_jx_j-x_k),\\ U_{K,p}(x)=&(-1)^p\sum_{\substack{L\subset K,\,|L|=p\\ \varepsilon_l=\pm1,\ l\in L}}\prod_{l\in L}w(\varepsilon_lx_l)\prod_{\substack{l,l'\in L\\ l<l'}}v(\varepsilon_lx_l+\varepsilon_{l'}x_{l'})v(-\varepsilon_lx_l-\varepsilon_{l'}x_{l'}-1)\\ &\times\prod_{\substack{l\in L\\ k\in K\setminus L}}v(\varepsilon_lx_l+x_k)v(\varepsilon_lx_l-x_k),\end{aligned}$$

with

$$v(z) = \frac{\mathrm{sh}\frac{\alpha}{2}(g+z)}{\mathrm{sh}(\frac{\alpha}{2}z)},$$

$$w(z) = \frac{\mathrm{sh}\frac{\alpha}{2}(g_0+z)}{\mathrm{sh}(\frac{\alpha}{2}z)}\,\frac{\mathrm{ch}\frac{\alpha}{2}(g_1+z)}{\mathrm{ch}(\frac{\alpha}{2}z)}\,\frac{\mathrm{sh}\frac{\alpha}{2}(g_2+\frac{1}{2}+z)}{\mathrm{sh}\frac{\alpha}{2}(\frac{1}{2}+z)}\,\frac{\mathrm{ch}\frac{\alpha}{2}(g_3+\frac{1}{2}+z)}{\mathrm{ch}\frac{\alpha}{2}(\frac{1}{2}+z)}.$$

The eigenvalues are determined by

$$E_r\,(y) = 2^r \sum_{\substack{J\subset\{1,\ldots,n\}\\ 0\leq |J|\leq r}} (-1)^{r-|J|}\Bigl(\prod_{j\in J}\mathrm{ch}(\alpha y_j) \sum_{r\leq l_1\leq\cdots\leq l_{r-|J|}\leq n} \mathrm{ch}(\alpha\rho_{l_1})\cdots\mathrm{ch}(\alpha\rho_{l_{r-|J|}})\Bigr),$$

with

$$\rho_j = (n-j)g + (g_0+\cdots+g_3)/2$$

($\rho = \sum_{j=1}^n \rho_j\, e_j$). In the above formulas $|J|$ represents the number of elements of $J\subset\{1,\ldots,n\}$, and we have used the conventions that empty products are equal to one, $U_{K,\,p}(x)\equiv 1$ if $p=0$, and that the second sum in $E_r(y)$ is equal to one if $|J|=r$.

For $r=1$ the Difference equation (3.1) reduces to a difference equation already considered by Koornwinder [10] and (for special parameters) Macdonald [11].

IV. RECURRENCE RELATIONS

To describe the recurrence relations for the multivariable Askey-Wilson polynomials it is convenient to introduce dual parameters $\hat{g}, \hat{g}_r$, which are related to the parameters g, g_r by

$$\hat{g} = g, \qquad \begin{pmatrix}\hat{g}_0\\ \hat{g}_1\\ \hat{g}_2\\ \hat{g}_3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&1&1&1\\ 1&1&-1&-1\\ 1&-1&1&-1\\ 1&-1&-1&1\end{pmatrix}\begin{pmatrix}g_0\\ g_1\\ g_2\\ g_3\end{pmatrix}. \tag{4.1}$$

Furthermore, instead of working with monic polynomials we will pass to a different normalization by introducing

$$P_\lambda(x) = c_\lambda^{-1} p_\lambda(x), \qquad c_\lambda = \hat{\Delta}^+(\rho+\lambda)/\hat{\Delta}^+(\rho), \tag{4.2}$$

where $\hat{\Delta}^+(x)$ is defined as in Eq. (2.5) but with the parameters g, g_r replaced by the dual parameters $\hat{g}, \hat{g}_r$. Similarly, we will also use the notation $\hat{p}_\lambda(x)$ and $\hat{P}_\lambda(x)$ ($=\hat{c}_\lambda^{-1}\hat{p}_\lambda(x)$ with $\hat{c}_\lambda = \Delta^+(\hat{\rho}+\lambda)/\Delta^+(\hat{\rho})$) for the corresponding dual polynomials (in which g, g_r again gets replaced by $\hat{g}, \hat{g}_r$).

It was conjectured by Macdonald [12] that the renormalized multivariable Askey-Wilson polynomials $P_\lambda(x)$ satisfy the relation

$$P_\lambda(\hat{\rho}+\mu) = \hat{P}_\mu(\rho+\lambda), \qquad \lambda,\mu\in\Lambda. \tag{4.3}$$

Recently, this conjecture has been proved by Cherednik [3] for special parameters (corresponding to reduced root systems) and in [6] for more general parameters subject to the self-duality condition

$$g_0 - g_1 - g_2 - g_3 = 0 \tag{4.4}$$

(implying $\hat{g}_r = g_r$).

The crucial point is now that the Difference equations (3.1) combined with Relation (4.3) give rise to a system of recurrence relations for the renormalized multivariable Askey-Wilson polynomials. These recurrence relations read explicitly

$$\hat{E}_r(x)\,P_\lambda(x) = \sum_{\substack{J\subset\{1,\ldots,n\},\,0\leq|J|\leq r\\ \varepsilon_j=\pm 1,\, j\in J;\ \lambda+e_{\varepsilon J}\in\Lambda}} \hat{U}_{J^c,\,r-|J|}(\rho+\lambda)\,\hat{V}_{\varepsilon J,\,J^c}(\rho+\lambda)\,P_{\lambda+e_{\varepsilon J}}(x), \tag{4.5}$$

$r = 1, \ldots, n$ (where the superscript hats have again been employed to indicate that parameters have been replaced by dual parameters).

To arrive at Eq. (4.5) one first substitutes $x = \rho + \lambda$ in the difference equation $\hat{D}_r \hat{P}_\mu = \hat{E}_r(\hat{\rho} + \mu)\hat{P}_\mu$ for the dual polynomials $\hat{P}_\mu$. Invoking of Relation (4.3) and using that $\hat{V}_{\varepsilon J,\, J^c}(\rho + \lambda) = 0$ if $\lambda + e_{\varepsilon J} \notin \Lambda$ then leads to Eq. (4.5) for $x = \hat{\rho} + \mu$ with $\mu \in \Lambda$. However, since Eq. (4.5) describes a relation between trigonometric polynomials, knowing that the relation is satisfied for all $\hat{\rho} + \mu$, $\mu \in \Lambda$, is in fact sufficient to conclude that equality must hold identically for all values of x.

V. ORTHONORMALIZATION

The difference operators D_r (3.2) are symmetric with respect to the inner product $\langle \cdot, \cdot \rangle_\Delta$ (2.3) and the functions $E_r(y)$, determining the eigenvalues in Eq. (3.1), separate the points $\rho + \lambda$, $\lambda \in \Lambda$ [4]. Hence, it follows that the eigenfunctions $p_\lambda(x)$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_\Delta$ (a priori the definition of the polynomials guarantees only that $\langle p_\mu, p_\lambda \rangle_\Delta = 0$ if $\mu < \lambda$).

It turns out that the Recurrence relations (4.5) may be employed to compute the normalization constants turning the polynomials into an orthonormal system. More precisely, if one works out both sides of the identity

$$\langle \hat{E}_r P_\lambda, P_{\lambda+\omega_r} \rangle_\Delta = \langle P_\lambda, \hat{E}_r P_{\lambda+\omega_r} \rangle_\Delta, \qquad \omega_r \equiv e_1 + \cdots + e_r,$$

using Recurrence relation (4.5) and the orthogonality of the polynomials, then one arrives at a relation between $\langle P_\lambda, P_\lambda \rangle_\Delta$ and $\langle P_{\lambda+\omega_r}, P_{\lambda+\omega_r} \rangle_\Delta$. By iterating this relation one can express $\langle P_\lambda, P_\lambda \rangle_\Delta$ in terms of $\langle 1, 1 \rangle_\Delta$ (which corresponds to $\lambda = 0$). Since the value of $\langle 1, 1 \rangle_\Delta$ is known from the work of Gustafson [7] (see also [8]), this solves the question of determining the orthonormalization constants. The answer, finally, reads [6]:

$$\langle p_\lambda, p_\lambda \rangle_\Delta = 2^n n!\, \hat{\Delta}^+(\rho + \lambda) \hat{\Delta}^-(\rho + \lambda), \tag{5.1}$$

with $\hat{\Delta}^+(x)$ taken the same as in Section IV and

$$\hat{\Delta}^-(x) = \prod_{1 \le j < k \le n} \hat{d}_v^-(x_j + x_k)\, \hat{d}_v^-(x_j - x_k) \prod_{1 \le j \le n} \hat{d}_w^-(x_j), \tag{5.2}$$

where

$$\begin{aligned}
\hat{d}_v^-(z) \;&=\; q^{\hat{g} z/2} \frac{(q^{(z+1)};\, q)_\infty}{(q^{(-\hat{g}+z+1)};\, q)_\infty}, \\
\hat{d}_w^-(z) &= q^{(\hat{g}_0 + \cdots + \hat{g}_3) z/2} \frac{(q^{(z+1)},\, -q^{(z+1)},\, q^{(1/2+z)},\, -q^{(1/2+z)};\, q)_\infty}{(q^{(-\hat{g}_0+z+1)},\, -q^{(-\hat{g}_1+z+1)},\, q^{(-\hat{g}_2+1/2+z)},\, -q^{(-\hat{g}_3+1/2+z)};\, q)_\infty}.
\end{aligned}$$

Remarks: i. The orthogonality of the multivariable Askey-Wilson polynomials was first proved by Koornwinder [10] using only the difference operator D_1. This proof is based on the continuity of $\langle p_\lambda, p_\mu \rangle_\Delta$ in the parameters and the fact that for generic parameters $E_1(\rho + \lambda) \neq E_1(\rho + \mu)$ if $\lambda \neq \mu$.

ii. The value for $\langle p_\lambda, p_\lambda \rangle_\Delta$ in (5.1) has first been conjectured by Macdonald [12] and was then proved by Cherednik for special parameters (corresponding to reduced root systems) using the representation theory of affine Hecke algebras [2]. Recently, Macdonald announced that Cherednik's Hecke-algebraic techniques can be extended to a proof of Formula (5.1) valid for general parameters [13] (see also [14]).

iii. Our derivation of the Recurrence relations (4.5) hinges on Relation (4.3). Since the proof of the latter relation presented in [6] only covers the self-dual case, to date our elementary proof of the Recurrence relations (4.5) (and consequently Formula (5.1)) is not complete unless the Self-duality condition (4.4) is satisfied.

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